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A state space formalism for piezothermoelasticity

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Abstract

A state space formalism for electrothermoelastic analysis of a linear piezoelectric body is developed. A novel feature of the formalism is that by proper grouping of the field variables and using matrix notations the three-dimensional equations of piezothermoelasticity are concisely formulated into a state equation and an output equation, which bear a striking resemblance to their elastic counterparts. The formalism is remarkably simple, with which one deals with only three vector quantities with the 13 independent electromechanical variables as their components and six submatrices that represent all the material constants for a piezoelectric material of the most general kind. In this work emphasis is placed on the state space formulation and solution to the generalized plane problem. Exact solutions for a piezoelectric half-space under a line of electromechanical loading and an infinite piezoelectric plate with an elliptical notch subjected to inplane loads are determined with relative ease. In many cases, piezoelectric solutions can be obtained directly from the corresponding elastic solutions by a simple replacement of the corresponding matrices on the basis of the formalism.

Keywords: Piezothermoelasticity; Piezoelectric materials; Electromechanical coupling; Generalized plane problems; State space formalism

1. Introduction

When a piezoelectric body is subjected to electromechanical loading, the electric and mechanical fields interact. In linear piezoelectricity the equations of elasticity are coupled to the equations of electrostatics through piezoelectric constants (Tiersten, 1969; Nowacki, 1975). Due to the electromechanical coupling and material anisotropy, electroelastic analysis is much more involved than its elastic counterpart. For problems in the Cartesian coordinate system it is natural to extend the Stroh formalism of anisotropic elasticity to include the piezoelectric effects. Such an extension was considered by Barnett and Lothe (1975) in studying the line charge–dislocation solution for a piezoelectric medium. Ting and his associate (Chung and Ting, 1995a,b, 1996; Ting, 1996) described an octet formalism for piezoelectric materials and applied it to the problem of a line force, a line dislocation, and a line charge in anisotropic piezoelectric spaces or wedges, and to a two-dimensional piezoelectric medium with an elliptic inclusion or hole. It is known that

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the Stroh formalism is intended for two-dimensional deformations of anisotropic elastic materials. The formalism is not well suited for problems in the cylindrical coordinate system.

The present work extends the state space formalism for anisotropic elasticity (Tarn, 2002a,b) to include the piezoelectric effects. In linear piezothermoelasticity the basic equations involve 54 piezothermoelastic constants for a piezoelectric material of the most general kind (Nye, 1957) and 13 independent electromechanical field variables (three displacements u_i, six stresses σ_{ij}, three electric displacement components D_i and one electric potential ϕ). The formulation would be intractable if the derivation is based on the equations as they stand. Guided by the experience with anisotropic elasticity, we group the field variables into two parts: $\boldsymbol{\tau}_2 = [\sigma_{12}, \sigma_{22}, \sigma_{23}, D_2]$, consisting of the components with one of the subscripts being 2, and the remaining components, $\boldsymbol{\tau}_1 = [\sigma_{13}, \sigma_{11}, \sigma_{33}, D_1, D_3]$. The choice of the field variables in a grouping depends on the problem under study. This particular choice is advantageous when the traction and electric charge are prescribed on the planes $x_2 =$ constant. By means of the judicious grouping and defining the matrix differential operators all the equations of piezothermoelasticity, including the constitutive equations, the equilibrium equations, the strain–displacement relations, the equations of electrostatics and the electric field–electric potential relations, are concisely formulated into two matrix differential equations in which the field variables appear in a direct and clear manner. The equations bear a remarkable resemblance to their elastic counterparts. Accordingly, derivation of the state equation and output equation follow the same line, leading to the same matrix state equation and output equation as those in anisotropic elasticity except for the entities of the matrices involved. The formalism brings in the stage $\mathbf{u} = [u_1, u_2, u_3, \phi]$, $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, and six sub-matrices of the piezothermoelastic constitutive matrix as the principal characters, the individual field variables and piezothermoelastic constants no longer play essential roles. Moreover, $\mathbf{u}$ and $\boldsymbol{\tau}_2$ are the only unknowns in the state equation. The analytic approach for anisotropic elasticity is carried over to piezothermoelasticity. When applied to generalized plane problems of a piezoelectric body, the homogeneous solution to the state equation takes the form of analytic functions of complex variables and the particular solution accounts for the effects of antiplane deformations, including extension, twisting and bending. The elastic solution is reproduced by setting the piezoelectric coefficients equal to zero.

The state space formalism is simple in concept and elegant in operation. In the present work the focus is on the basic formulation in the state space setting and attention is paid to the general solution to the generalized plane problems. For illustration, the problems of half-space and a plate with a notch are studied. Exact solutions for a piezoelectric half-space under a line of electromechanical loading and an infinite piezoelectric plate with an elliptical hole subjected to uniform extension are determined with relative ease.

2. State space formulation

We consider the piezoelectric material of the most general kind. The linear piezothermoelastic constitutive equations written out in full are (Nye, 1957; Nowacki, 1975)

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{23}\\\sigma_{13}\\\sigma_{12}\\D_1\\D_2\\D_3\end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & e_{11} & e_{21} & e_{31}\\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} & e_{12} & e_{22} & e_{32}\\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} & e_{13} & e_{23} & e_{33}\\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} & e_{14} & e_{24} & e_{34}\\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} & e_{15} & e_{25} & e_{35}\\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} & e_{16} & e_{26} & e_{36}\\ e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} & -\kappa_{11} & -\kappa_{12} & -\kappa_{13}\\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26} & -\kappa_{12} & -\kappa_{22} & -\kappa_{23}\\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} & -\kappa_{13} & -\kappa_{23} & -\kappa_{33} \end{bmatrix} \begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\2\varepsilon_{23}\\2\varepsilon_{13}\\2\varepsilon_{12}\\-E_1\\-E_2\\-E_3\end{bmatrix} - \begin{bmatrix}\beta_1\\\beta_2\\\beta_3\\\beta_4\\\beta_5\\\beta_6\\-p_1^{\varepsilon}\\-p_2^{\varepsilon}\\-p_3^{\varepsilon}\end{bmatrix} T, \tag{1}$$

where σ_{ij} and ε_{ij} are the stress and strain tensors, D_i and E_i are the electric displacement and electric field vectors, T is the temperature field, c_{ij} the 21 elastic constants measured at a constant electric field and constant temperature, κ_{ij} the *permittivity constants* measured at constant strain and constant temperature, e_{ij} the *piezoelectric constants* measured at constant temperature, β_i the thermal coefficients measured at constant electric field, p_i^{ε} the *pyroelectric constants* measured at constant strain.

Eq. (1) is the constitutive relations for piezoelectric materials of all the 32 crystal classes (Nye, 1957) and piezoelectric ceramics. There are 54 independent piezothermoelastic constants in total for a piezoelectric material of the most general kind. The piezoelectric body may exhibits special type of material anisotropy, including among others, the monoclinic system of class 2 polarized in the x_2-direction by setting $c_{ij} = 0$ $(i = 1, 2, 3, 5,\ j = 4, 6)$, $e_{ij} = 0$ $(i = 1, 3,\ j = 1, 2, 3, 5)$, $e_{24} = 0$, $\kappa_{12} = \kappa_{23} = \beta_4 = \beta_6 = p_1^{\sigma} = p_3^{\sigma} = 0$; the orthorhombic system of class $mm2$ polarized in the x_2-direction by setting in addition, $c_{i5} = 0$ $(i = 1, 2, 3)$, $c_{46} = e_{14} = e_{25} = e_{36} = \kappa_{13} = \beta_5 = 0$, $c_{11} = c_{33}$, $c_{12} = c_{23}$, $c_{44} = c_{66}$, $e_{21} = e_{23}$, $e_{16} = e_{34}$, $\kappa_{11} = \kappa_{33}$, $\beta_1 = \beta_3$. In particular, Eq. (1) includes the constitutive equations of anisotropic elastic materials as a special case with $e_{ij} = 0$.

The strain–displacement relations are

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2, \tag{2}$$

where u_i denotes the displacement components, a comma indicates differentiation with respect to the suffix variable.

The equations of motion are

$$\sigma_{ij,j} + F_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \tag{3}$$

where F_i denotes the body force components.

The equations of electrostatics without the free charge are

$$D_{i,i} = 0 \tag{4}$$

and the relations between the electric field and the electric potential are

$$E_i = -\phi_{,i}, \tag{5}$$

where ϕ is the electric potential.

The formulation could be greatly simplified if the field variables are grouped properly. For the problems in the Cartesian coordinate system, if the x_2-axis is pointed in the thickness direction, the traction vector on the planes $x_2 =$ constant are $(\sigma_{12}, \sigma_{22}, \sigma_{23})$, thus the traction boundary conditions and the interfacial continuity conditions are directly associated with them. Furthermore, the normal electric displacement D_2 are also associated with these conditions. With this in mind, we group the field variables into two parts: one consists of the components associated with the subscript 2, another consists of the remaining components, and rewrite Eq. (1) as

$$\begin{bmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{12}^{\mathrm{T}} & \mathbf{C}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\gamma}_1 \\ \boldsymbol{\gamma}_2 \end{bmatrix} - \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix} T, \tag{6}$$

where

$$\boldsymbol{\tau}_1 = [\,\sigma_{13} \quad \sigma_{11} \quad \sigma_{33} \quad D_1 \quad D_3\,]^{\mathrm{T}}, \qquad \boldsymbol{\tau}_2 = [\,\sigma_{12} \quad \sigma_{22} \quad \sigma_{23} \quad D_2\,]^{\mathrm{T}},$$

$$\boldsymbol{\gamma}_1 = [\,2\varepsilon_{13} \quad \varepsilon_{11} \quad \varepsilon_{33} \quad -E_1 \quad -E_3\,]^{\mathrm{T}}, \qquad \boldsymbol{\gamma}_2 = [\,2\varepsilon_{12} \quad \varepsilon_{22} \quad 2\varepsilon_{23} \quad -E_2\,]^{\mathrm{T}},$$

$$\mathbf{C}_{11} = \begin{bmatrix} c_{55} & c_{15} & c_{35} & e_{15} & e_{35} \\ c_{15} & c_{11} & c_{13} & e_{11} & e_{31} \\ c_{35} & c_{13} & c_{33} & e_{13} & e_{33} \\ e_{15} & e_{11} & e_{13} & -\kappa_{11} & -\kappa_{13} \\ e_{35} & e_{31} & e_{33} & -\kappa_{13} & -\kappa_{33} \end{bmatrix}, \quad \mathbf{C}_{12} = \begin{bmatrix} c_{56} & c_{25} & c_{45} & e_{25} \\ c_{16} & c_{12} & c_{14} & e_{21} \\ c_{36} & c_{23} & c_{34} & e_{23} \\ e_{16} & e_{12} & e_{14} & -\kappa_{12} \\ e_{36} & e_{32} & e_{34} & -\kappa_{23} \end{bmatrix}, \quad \boldsymbol{\beta}_1 = \begin{bmatrix} \beta_5 \\ \beta_1 \\ \beta_3 \\ -p_1^{\varepsilon} \\ -p_3^{\varepsilon} \end{bmatrix},$$

$$\mathbf{C}_{22} = \begin{bmatrix} c_{66} & c_{26} & c_{46} & e_{26} \\ c_{26} & c_{22} & c_{24} & e_{22} \\ c_{46} & c_{24} & c_{44} & e_{24} \\ e_{26} & e_{22} & e_{24} & -\kappa_{22} \end{bmatrix}, \quad \boldsymbol{\beta}_2 = \begin{bmatrix} \beta_6 \\ \beta_2 \\ \beta_4 \\ -p_2^{\varepsilon} \end{bmatrix}.$$

The strain–displacement relations may be expressed as

$$\begin{bmatrix} \boldsymbol{\gamma}_1 \\ \boldsymbol{\gamma}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{L}_1\mathbf{u} \\ \mathbf{L}_2\mathbf{u} \end{bmatrix} + \partial_2 \begin{bmatrix} \mathbf{0} \\ \mathbf{u} \end{bmatrix},$$

where ∂_i stands for the partial derivative with respect to x_i, and

$$\mathbf{u} = [u_1 \quad u_2 \quad u_3 \quad \phi]^{\mathrm{T}}, \quad \mathbf{L}_1 = \mathbf{K}_1\partial_1 + \mathbf{K}_2\partial_3, \quad \mathbf{L}_2 = \mathbf{K}_3\partial_1 + \mathbf{K}_4\partial_3,$$

$$\mathbf{K}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{K}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The strain and the electric field vector are intermediate variables. Expressing them in terms of the stress and the electric potential by substituting Eq. (3) into Eq. (2), one obtains

$$\begin{bmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \end{bmatrix} = \begin{bmatrix} (\mathbf{C}_{11}\mathbf{L}_1 + \mathbf{C}_{12}\mathbf{L}_2)\mathbf{u} \\ (\mathbf{C}_{12}^{\mathrm{T}}\mathbf{L}_1 + \mathbf{C}_{22}\mathbf{L}_2)\mathbf{u} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{12}\partial_2\mathbf{u} \\ \mathbf{C}_{22}\partial_2\mathbf{u} \end{bmatrix} - \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix} T. \tag{7}$$

On defining the $\mathbf{K}_i$ matrices and the differential operators $\mathbf{L}_1$ and $\mathbf{L}_2$, the equations of motion and the equation of electrostatics can be written in a single matrix equation as

$$\partial_2\boldsymbol{\tau}_2 + \mathbf{L}_1^{\mathrm{T}}\boldsymbol{\tau}_1 + \mathbf{L}_2^{\mathrm{T}}\boldsymbol{\tau}_2 + \mathbf{F} = \mathbf{K}_\rho \frac{\partial^2}{\partial t^2}\mathbf{u}, \tag{8}$$

where

$$\mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ 0 \end{bmatrix}, \qquad \mathbf{K}_\rho = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Eqs. (7) and (8) embrace the three-dimensional equations of piezothermoelasticity in full. They bear a remarkable resemblance to their elastic counterparts (Tarn, 2002a), differing only in the entities of the matrices due to the piezoelectric effects. With the basic equations so expressed, the individual components of the field variables and constitutive matrices are no longer in view—they are replaced by $\mathbf{u}$, $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, $\mathbf{C}_{\alpha\beta}$ and $\boldsymbol{\beta}_\alpha$ $(\alpha, \beta = 1, 2)$ which play the principal roles hereafter. Note that it is possible to group the field variables in other ways. For a new grouping only the matrices $\mathbf{C}_{\alpha\beta}$, $\mathbf{K}_i$ and $\mathbf{K}_\rho$ need to be adjusted accordingly.

Following the derivation for anisotropic elasticity (Tarn, 2002a), choosing $[\mathbf{u}, \boldsymbol{\tau}_2]^{\mathrm{T}}$ as the state vector, one can write out at once the state equation and the output equation of piezothermoelasticity as follows:

$$\frac{\partial}{\partial x_2}\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\tau}_2\end{bmatrix} = \begin{bmatrix}\mathbf{D}_{11} & \mathbf{C}_{22}^{-1}\\ \mathbf{D}_{21} & \mathbf{D}_{11}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\tau}_2\end{bmatrix} + \begin{bmatrix}\mathbf{C}_{22}^{-1}\boldsymbol{\beta}_2\\ \mathbf{L}_1^{\mathrm{T}}\tilde{\boldsymbol{\beta}}_1\end{bmatrix}T - \begin{bmatrix}\mathbf{0}\\ \mathbf{F}\end{bmatrix} + \frac{\partial^2}{\partial t^2}\begin{bmatrix}\mathbf{0}\\ \mathbf{K}_\rho\mathbf{u}\end{bmatrix}, \tag{9}$$

$$\boldsymbol{\tau}_1 = [\,\widetilde{\mathbf{C}}_{11}\mathbf{L}_1 \quad \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\,]\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\tau}_2\end{bmatrix} - \tilde{\boldsymbol{\beta}}_1 T, \tag{10}$$

where

$$\mathbf{D}_{11} = -\mathbf{C}_{22}^{-1}\mathbf{C}_{12}^{\mathrm{T}}\mathbf{L}_1 - \mathbf{L}_2, \qquad \mathbf{D}_{21} = -\mathbf{L}_1^{\mathrm{T}}\widetilde{\mathbf{C}}_{11}\mathbf{L}_1,$$

$$\widetilde{\mathbf{C}}_{11} = \mathbf{C}_{11} - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\mathbf{C}_{12}^{\mathrm{T}}, \qquad \tilde{\boldsymbol{\beta}}_1 = \boldsymbol{\beta}_1 - \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\boldsymbol{\beta}_2.$$

It is readily shown, in the same way as for anisotropic elasticity, that the stiffness-based and compliance-based formalisms are completely equivalent.

3. Generalized plane problems

Let us focus on the static response of the piezoelectric body under electromechanical loadings that do not vary in the x_3-axis. The problem is referred to as the generalized plane problem, including generalized plane strain and generalized torsion of the body. For this class of problem the stress and strain fields are independent of x_3. The displacement field is given by

$$u_1 = u - b_1x_3^2/2 - \vartheta x_2x_3 - \omega_3x_2 + \omega_2x_3 + u_0, \tag{11}$$

$$u_2 = v - b_2x_3^2/2 + \vartheta x_1x_3 + \omega_3x_1 - \omega_1x_3 + v_0, \tag{12}$$

$$u_3 = w + (b_1x_1 + b_2x_2 + \varepsilon)x_3 - \omega_2x_1 + \omega_1x_2 + w_0, \tag{13}$$

where u, v, w are unknown functions of x_1 and x_2; ω_1, ω_2, ω_3 and u_0, v_0, w_0 are constants characterizing the rigid body displacements. The constant ε is a uniform extension, ϑ is associated with the curvature due to twisting, b_1 and b_2 are associated with the curvatures due to bending.

On substituting Eqs. (11)–(13) in Eqs. (9) and (10), the state equation and the output equation for the generalized plane problem read

$$\frac{\partial}{\partial x_2}\begin{bmatrix}\tilde{\mathbf{u}}\\ \boldsymbol{\tau}_2\end{bmatrix} = \begin{bmatrix}-\mathbf{C}_{22}^{-1}\mathbf{A}_1\partial_1 & \mathbf{C}_{22}^{-1}\\ -\mathbf{A}_2\partial_{11} & -\mathbf{A}_1^{\mathrm{T}}\mathbf{C}_{22}^{-1}\partial_1\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{u}}\\ \boldsymbol{\tau}_2\end{bmatrix} - \begin{bmatrix}\mathbf{p}_1\\ \mathbf{p}_2\end{bmatrix} + \begin{bmatrix}\mathbf{C}_{22}^{-1}\boldsymbol{\beta}_2\\ \mathbf{L}_1^{\mathrm{T}}\tilde{\boldsymbol{\beta}}_1\end{bmatrix}T - \begin{bmatrix}\mathbf{0}\\ \mathbf{f}\end{bmatrix}, \tag{14}$$

$$\boldsymbol{\tau}_1 = [\,\widetilde{\mathbf{C}}_{11}\mathbf{K}_1\partial_1 \quad \mathbf{C}_{12}\mathbf{C}_{22}^{-1}\,]\begin{bmatrix}\tilde{\mathbf{u}}\\ \boldsymbol{\tau}_2\end{bmatrix} + \widetilde{\mathbf{C}}_{11}[(\varepsilon + b_1x_1 + b_2x_2)\mathbf{k}_1 - \vartheta x_2\mathbf{k}_2] - \tilde{\boldsymbol{\beta}}_1 T, \tag{15}$$

where $T = T(x_1, x_2)$, the body force F_3 should not present, and

$$\tilde{\mathbf{u}} = [\,u \quad v \quad w \quad \phi\,]^{\mathrm{T}}, \qquad \mathbf{f} = [\,F_1 \quad F_2 \quad 0 \quad 0\,]^{\mathrm{T}},$$

$$\mathbf{A}_1 = \mathbf{C}_{12}^{\mathrm{T}}\mathbf{K}_1 + \mathbf{C}_{22}\mathbf{K}_3, \qquad \mathbf{A}_2 = \mathbf{K}_1^{\mathrm{T}}\widetilde{\mathbf{C}}_{11}\mathbf{K}_1,$$

$$\mathbf{p}_1 = \mathbf{C}_{22}^{-1}\mathbf{C}_{12}^{\mathrm{T}}[(\varepsilon + b_1x_1 + b_2x_2)\mathbf{k}_1 - \vartheta x_2\mathbf{k}_2] + \vartheta x_1\mathbf{k}_3, \qquad \mathbf{p}_2 = b_1[\,\tilde{c}_{13} \quad 0 \quad \tilde{c}_{35} \quad \tilde{e}_{13}\,]^{\mathrm{T}},$$

$$\widetilde{\mathbf{C}}_{11} = \begin{bmatrix} \tilde{c}_{55} & \tilde{c}_{15} & \tilde{c}_{35} & \tilde{e}_{15} & \tilde{e}_{35} \\ \tilde{c}_{15} & \tilde{c}_{11} & \tilde{c}_{13} & \tilde{e}_{11} & \tilde{e}_{31} \\ \tilde{c}_{35} & \tilde{c}_{13} & \tilde{c}_{33} & \tilde{e}_{13} & \tilde{e}_{33} \\ \tilde{e}_{15} & \tilde{e}_{11} & \tilde{e}_{13} & -\tilde{\kappa}_{11} & -\tilde{\kappa}_{13} \\ \tilde{e}_{35} & \tilde{e}_{31} & \tilde{e}_{33} & -\tilde{\kappa}_{13} & -\tilde{\kappa}_{33} \end{bmatrix}, \quad \mathbf{k}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{k}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{k}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Let us seek the homogeneous solution of Eq. (14) in the form

$$\tilde{\mathbf{u}} = \mathbf{U}F(z), \qquad \boldsymbol{\tau}_2 = \mathbf{S}F'(z), \tag{16}$$

where $\mathbf{U}$ and $\mathbf{S}$ are constant vectors, each has four components; $F(z)$ is an unknown function, $F'(z) = \mathrm{d}F(z)/\mathrm{d}z$, $z = x_1 + px_2$, p is a constant parameter to be determined.

Substituting Eq. (16) in Eq. (14) yields the eigen relation:

$$\begin{bmatrix} -\mathbf{C}_{22}^{-1}\mathbf{A}_1 & \mathbf{C}_{22}^{-1} \\ -\mathbf{A}_2 & -\mathbf{A}_1^{\mathrm{T}}\mathbf{C}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{S} \end{bmatrix} = p \begin{bmatrix} \mathbf{U} \\ \mathbf{S} \end{bmatrix}, \tag{17}$$

where p is the eigenvalue, $[\mathbf{U}, \mathbf{S}]^{\mathrm{T}}$ is the eigenvector. For a given material the eigenvalues and eigenvectors can be easily determined using *Mathematica* or MATLAB.

Expressing $\mathbf{S}$ in terms of $\mathbf{U}$ using Eq. $(17)_1$ gives

$$\mathbf{S} = (\mathbf{A}_1 + p\mathbf{C}_{22})\mathbf{U}. \tag{18}$$

Substituting Eq. (18) in Eq. $(17)_2$ leads to

$$[\mathbf{A}_3 + p(\mathbf{A}_1 + \mathbf{A}_1^{\mathrm{T}}) + p^2\mathbf{C}_{22}]\mathbf{U} = \mathbf{0}, \tag{19}$$

where

$$\mathbf{A}_3 = \mathbf{K}_1^{\mathrm{T}}\mathbf{C}_{11}\mathbf{K}_1 + \mathbf{K}_1^{\mathrm{T}}\mathbf{C}_{12}\mathbf{K}_3 + \mathbf{K}_3^{\mathrm{T}}\mathbf{C}_{12}^{\mathrm{T}}\mathbf{K}_1 + \mathbf{K}_3^{\mathrm{T}}\mathbf{C}_{22}\mathbf{K}_3.$$

Non-trivial solution to Eq. (19) exists if and only if the determinant of the coefficient matrix vanish,

$$|\mathbf{A}_3 + p(\mathbf{A}_1 + \mathbf{A}_1^{\mathrm{T}}) + p^2\mathbf{C}_{22}| = 0, \tag{20}$$

in which the explicit expressions of the matrices $\mathbf{A}_i$ are

$$\mathbf{A}_1 = \begin{bmatrix} c_{16} & c_{66} & c_{56} & e_{16} \\ c_{12} & c_{26} & c_{25} & e_{12} \\ c_{14} & c_{46} & c_{45} & e_{14} \\ e_{21} & e_{26} & e_{25} & -\kappa_{12} \end{bmatrix}, \qquad \mathbf{A}_3 = \begin{bmatrix} c_{11} & c_{16} & c_{15} & e_{11} \\ c_{16} & c_{66} & c_{56} & e_{16} \\ c_{15} & c_{56} & c_{55} & e_{15} \\ e_{11} & e_{16} & e_{15} & -\kappa_{11} \end{bmatrix}.$$

In the absence of the piezoelectric effects, setting $e_{ij} = 0$ reduces Eq. (20) to the sextic equation of Stroh and Eq. (17) to the eigen relation posed in the Stroh formalism for anisotropic elasticity.

It can be shown that the p cannot be real by virtue of the positive definiteness of the electric enthalpy, and there are four pairs of complex conjugate p. Denoting the eigenvalues and the associated eigenvectors by

$$p_k = a_k + \mathrm{i}b_k, \quad p_{k+4} = \bar{p}_k = a_k - \mathrm{i}b_k \quad (b_k > 0), \tag{21}$$

$$\mathbf{U}_{k+4} = \overline{\mathbf{U}}_k, \quad \mathbf{S}_{k+4} = \overline{\mathbf{S}}_k \quad (k = 1, 2, 3, 4), \tag{22}$$

where 'i' is the imaginary number, a_k and b_k are real, there follow

$$\tilde{\mathbf{u}} = 2\mathrm{Re}\left\{ \sum_{k=1}^{4} \mathbf{U}_k F_k(z_k) \right\}, \tag{23}$$

$$\boldsymbol{\tau}_1 = 2\mathrm{Re}\left\{ \sum_{k=1}^{4} (\mathbf{A}_4 + p_k \mathbf{C}_{12}) \mathbf{U}_k F_k'(z_k) \right\}, \tag{24}$$

$$\boldsymbol{\tau}_2 = 2\mathrm{Re}\left\{ \sum_{k=1}^{4} (\mathbf{A}_1 + p_k \mathbf{C}_{22}) \mathbf{U}_k F_k'(z_k) \right\}, \tag{25}$$

where the $\mathbf{U}_k$ are the eigenvectors associated with p_k, and

$$\mathbf{A}_4 = \mathbf{C}_{11}\mathbf{K}_1 + \mathbf{C}_{12}\mathbf{K}_3 = \begin{bmatrix} c_{15} & c_{56} & c_{55} & e_{15} \\ c_{11} & c_{16} & c_{15} & e_{11} \\ c_{13} & c_{36} & c_{35} & e_{13} \\ e_{11} & e_{16} & e_{15} & -\kappa_{11} \\ e_{31} & e_{36} & e_{35} & -\kappa_{13} \end{bmatrix}.$$

For a uniform temperature change ΔT and a constant body force $\mathbf{f}$, the particular solution to the state equation is easily found to be

$$\tilde{\mathbf{u}} = \mathbf{a}_1 x_1^2/2 + \mathbf{a}_2 x_1 x_2 + \mathbf{a}_3 x_2^2/2, \tag{26}$$

$$\boldsymbol{\tau}_2 = \varepsilon \mathbf{C}_{12}^{\mathrm{T}} \mathbf{k}_1 - \boldsymbol{\beta}_2 \Delta T, \tag{27}$$

$$\boldsymbol{\tau}_1 = \varepsilon \mathbf{C}_{11} \mathbf{k}_1 - \vartheta \widetilde{\mathbf{C}}_{11} \mathbf{k}_2 x_2 + \mathbf{C}_{11}[(\mathbf{K}_1 \mathbf{a}_1 + b_1 \mathbf{k}_1) x_1 + (\mathbf{K}_1 \mathbf{a}_2 + b_2 \mathbf{k}_1) x_2] - \boldsymbol{\beta}_1 \Delta T, \tag{28}$$

in which $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are determined from

$$\mathbf{K}_1^{\mathrm{T}} \widetilde{\mathbf{C}}_{11} \mathbf{K}_1 \mathbf{a}_1 = -\mathbf{p}_2 - \mathbf{f}, \tag{29}$$

$$\mathbf{A}_1 \mathbf{a}_1 + \mathbf{C}_{22} \mathbf{a}_2 = -b_1 \mathbf{C}_{12}^{\mathrm{T}} \mathbf{k}_1 - \vartheta \mathbf{C}_{22} \mathbf{k}_1, \tag{30}$$

$$\mathbf{A}_1 \mathbf{a}_2 + \mathbf{C}_{22} \mathbf{a}_3 = -b_2 \mathbf{C}_{12}^{\mathrm{T}} \mathbf{k}_1 + \vartheta \mathbf{C}_{12}^{\mathrm{T}} \mathbf{k}_2. \tag{31}$$

The general solution is obtained by superposing Eqs. (23)–(25) and (26)–(28) along with Eqs. (11)–(13) as

$$\mathbf{u} = 2\mathrm{Re}\left\{ \sum_{k=1}^{4} \mathbf{U}_k F_k(z_k) \right\} + \mathbf{a}_1 x_1^2/2 + \mathbf{a}_2 x_1 x_2 + \mathbf{a}_3 x_2^2/2 + \hat{\mathbf{u}}, \tag{32}$$

$$\begin{aligned} \boldsymbol{\tau}_1 = {} & 2\mathrm{Re}\left\{ \sum_{k=1}^{4} (\mathbf{A}_4 + p_k \mathbf{C}_{12}) \mathbf{U}_k F_k'(z_k) \right\} + \varepsilon \mathbf{C}_{11} \mathbf{k}_1 - \vartheta \widetilde{\mathbf{C}}_{11} \mathbf{k}_2 x_2 \\ & + \widetilde{\mathbf{C}}_{11}[(\mathbf{K}_1 \mathbf{a}_1 + b_1 \mathbf{k}_1) x_1 + (\mathbf{K}_1 \mathbf{a}_2 + b_2 \mathbf{k}_1) x_2] - \boldsymbol{\beta}_1 \Delta T, \end{aligned} \tag{33}$$

$$\boldsymbol{\tau}_2 = 2\mathrm{Re}\left\{ \sum_{k=1}^{4} (\mathbf{A}_1 + p_k \mathbf{C}_{22}) \mathbf{U}_k F_k'(z_k) \right\} + \varepsilon \mathbf{C}_{12}^{\mathrm{T}} \mathbf{k}_1 - \boldsymbol{\beta}_2 \Delta T, \tag{34}$$

in which the complex potentials $F_k(z_k)$ for a specific problem are to be determined, and

$$\hat{\mathbf{u}} = \begin{bmatrix} -b_1 x_3^2/2 - \vartheta x_2 x_3 - \omega_3 x_2 + \omega_2 x_3 + u_0 \\ -b_2 x_3^2/2 + \vartheta x_1 x_3 + \omega_3 x_1 - \omega_1 x_3 + v_0 \\ (b_1 x_1 + b_2 x_2 + \varepsilon) x_3 - \omega_2 x_1 + \omega_1 x_2 + w_0 \end{bmatrix}.$$

Except for plane deformations in which the b_1, b_2, ε and ϑ are known to be zero in advance, they must be determined by the end conditions that require the stress resultants over the cross-section Ω reduce to an axial force P_z, a torque M_t, and bi-axial bending moments M_1, M_2:

$$\int_{\Omega} (\mathbf{H}_1 \boldsymbol{\tau}_1 + \mathbf{H}_2 \boldsymbol{\tau}_2)\, \mathrm{d}x_1\, \mathrm{d}x_2 = \mathbf{P}, \tag{35}$$

where

$$\mathbf{H}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & -x_1 & 0 & 0 \\ -x_2 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{H}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} P_z \\ M_1 \\ M_2 \\ M_t \end{bmatrix}.$$

As there is a one to one correspondence between the b_1, b_2, ε, ϑ and the prescribed loads through Eq. (35), these constants may be regarded as known a priori. It is noteworthy that the general solution to the generalized plane problems of a piezoelectric body closely resembles its elastic counterpart (Tarn, 2002a), differing only in the entities of the matrices involved.

4. Half-space under a line of electromechanical loading

The stress distribution in an elastic half-space under line loads or line dislocations is a classical problem in anisotropic elasticity (Lekhnitskii, 1981). Here the analysis is extended to a piezoelectric half-space subjected to a line of electromechanical loading and a uniform temperature change. The solution makes use of the Cauchy integral. It follows essentially the approach given in Section 28 of Lekhnitskii's monograph.

The mechanical boundary conditions of the problem are

$$\sigma_{22} = N(x_1), \quad \sigma_{12} = T(x_1), \quad \sigma_{23} = 0 \qquad \text{on } x_2 = 0. \tag{36}$$

The electric boundary conditions are such that either the electric charge or the voltage is prescribed:

$$D_2 = f(x_1) \quad \text{or} \quad \phi = g(x_1) \qquad \text{on } x_2 = 0, \tag{37}$$

where $N(x_1)$, $T(x_1)$, $f(x_1)$, and $g(x_1)$ are prescribed functions of x_1.

In case the traction and the electric charge are prescribed, the boundary conditions are particularly simple in the present context. They are

$$\boldsymbol{\tau}_2(x_1) = [\, T(x_1) \quad N(x_1) \quad 0 \quad f(x_1)\,]^{\mathrm{T}} \quad \text{on } x_2 = 0. \tag{38}$$

Likewise, the displacement and potential prescribed boundary conditions are expressible in terms of $\mathbf{u}$. When the traction and electric voltage are prescribed, the relevant components can be extracted from the state vector or may be derived from a new grouping from the beginning.

For illustration, we consider the half-space subjected to the boundary conditions prescribed by Eq. (38). Imposing Eq. (38) on the general solution yields the following conditions for the complex potentials $F_k'(z_k)$:

$$2\mathrm{Re}\left\{ \sum_{k=1}^{4} (\mathbf{A}_1 + p_k \mathbf{C}_{22}) \mathbf{U}_k F_k'(x_1) \right\} = \boldsymbol{\tau}_2(x_1) - \varepsilon \mathbf{C}_{12}^{\mathrm{T}} \mathbf{k}_1 + \boldsymbol{\beta}_2 \Delta T. \tag{39}$$

Applying the Cauchy integral formula for analytic functions of complex variables to Eq. (39) yields

$$\sum_{k=1}^{4} (\mathbf{A}_1 + p_k \mathbf{C}_{22}) \mathbf{U}_k F_k'(z) = -\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\boldsymbol{\tau}_2(x_1) - \varepsilon \mathbf{C}_{12}^{\mathrm{T}} \mathbf{k}_1 + \boldsymbol{\beta}_2 \Delta T}{x_1 - z} \mathrm{d}x_1. \tag{40}$$

This is a system of four equations for four unknown $F'_k(z)$. On solving them and replacing the variable z in F'_k by z_k to obtain $F'_k(z_k)$, it is straightforward to determine the electromechanical field using Eqs. (32)–(35).

We note in passing that the solution for an anisotropic elastic half-space subjected to a line load are recovered by setting $e_{ij} = 0$ and $\Delta T = 0$. More importantly, as the state equations in anisotropic elasticity and piezothermoelasticity are identical in form, in case the boundary conditions of a piezoelectric problem are also in the same matrix form as their elastic counterparts, (for example, in the absence of electric variables the traction boundary conditions of elasticity are also expressed by $\boldsymbol{\tau}_2$ for this problem), the piezoelectric solution can be obtained directly from the corresponding elastic solution by a proper replacement of matrices. Indeed, the solution presented herein can be written out from the solution for the elastic half-space subjected to a line load. The same is true when **u** is prescribed.

5. Extension of the piezoelectric plate with an elliptical hole

Numerous solutions to the notch problems of elastic materials have been presented and documented in monographs (see, for examples, Savin, 1961; Lekhnitskii, 1968, 1981). It is known that determination of the analytic solution to an anisotropic elastic plate with a hole hinges on the existence of the conformal mapping functions that transform the exterior of the hole onto the exterior of a unit circle for three complex variables. It has been shown (Wang and Tarn, 1993) that the conformal mapping in the entire region outside the unit circle is in general possible only for an elliptical hole, approximate solutions for non-elliptical hole or rigid inclusion are also studied therein. For piezoelectric materials there are four complex variables $z_k = x_1 + p_k x_2$ $(k = 1, 2, 3, 4)$, it is not possible to transform the hole of given shape onto a unit circle for all the z_k except for an elliptical hole.

Various piezothermoelastic problems of an elliptical hole or inclusion can be solved within the context. As an illustration, we consider the electromechanical field in an infinite piezoelectric plate with an elliptical hole under uniform extension at infinity.

For the problem under consideration the mechanical boundary conditions at the hole boundary are traction-free, the electric boundary conditions are such that the normal electric displacement at the boundary is zero. When the plate is subjected to a remote uniform tension σ_0 in the x_1-direction, the conditions at infinity are

$$\sigma_{11} = \sigma_0, \quad \sigma_{12} = \sigma_{13} = \sigma_{22} = \sigma_{23} = \sigma_{33} = 0, \quad \phi = \phi_0 = \text{constant}. \tag{41}$$

The notch problem under remote loading can be transformed to one with loading on the hole boundary by superposing the electromechanical field in an infinite piezoelectric plate and the auxiliary one due to applying to the contour of the hole a negative traction and normal electric displacement that are derived from the infinite plate solution. This makes the hole boundary traction-free and normal electric displacement zero and the boundary conditions at infinity are satisfied as well, thus yielding the solution to the original problem.

The electromechanical field in an infinite piezoelectric plate subjected to uniform tension is simply

$$\sigma_{11} = \sigma_0, \quad \sigma_{12} = \sigma_{13} = \sigma_{22} = \sigma_{23} = \sigma_{33} = 0; \tag{42}$$

$$\phi = \phi_0, \quad D_1 = d_{11}\sigma_0, \quad D_2 = d_{21}\sigma_0, \quad D_3 = d_{31}\sigma_0, \tag{43}$$

where d_{ij} are the coefficients of the converse piezoelectric effect measured at constant temperature. In the present notations the electromechanical field is

$$\boldsymbol{\tau}_1 = \sigma_0[0 \quad 1 \quad 0 \quad d_{11} \quad d_{31}]^{\mathrm{T}}, \qquad \boldsymbol{\tau}_2 = \sigma_0[0 \quad 0 \quad 0 \quad d_{21}]^{\mathrm{T}}. \tag{44}$$

It can be shown by using Eqs. (19), (24) and (25) that the traction components t_i and the normal electric displacement D_n on the hole boundary can be expressed in terms of the complex potential as follows:

$$t_1 = (\boldsymbol{\tau}_1^{\mathrm{T}}\mathbf{K}_1 + \boldsymbol{\tau}_2^{\mathrm{T}}\mathbf{K}_3)\mathbf{n}_\sigma = -2\mathrm{Re}\left\{\sum_{k=1}^{4} p_k\mathbf{U}_k^{\mathrm{T}}(\mathbf{A}_1^{\mathrm{T}} + p_k\mathbf{C}_{22})F_k'(z_k)\right\}\mathbf{n}_\sigma, \tag{45}$$

$$t_2 = \boldsymbol{\tau}_2^{\mathrm{T}}\mathbf{n}_\sigma = 2\mathrm{Re}\left\{\sum_{k=1}^{4} \mathbf{U}_k^{\mathrm{T}}(\mathbf{A}_1^{\mathrm{T}} + p_k\mathbf{C}_{22})F_k'(z_k)\right\}\mathbf{n}_\sigma, \tag{46}$$

$$t_3 = (\boldsymbol{\tau}_1^{\mathrm{T}}\mathbf{K}_2 + \boldsymbol{\tau}_2^{\mathrm{T}}\mathbf{K}_4)\mathbf{n}_\sigma = 2\mathrm{Re}\left\{\sum_{k=1}^{4} \mathbf{U}_k^{\mathrm{T}}(\mathbf{A}_5 + p_k\mathbf{A}_6)F_k'(z_k)\right\}\mathbf{n}_\sigma, \tag{47}$$

$$D_n = (\boldsymbol{\tau}_1^{\mathrm{T}}\mathbf{K}_5 + \boldsymbol{\tau}_2^{\mathrm{T}}\mathbf{K}_6)\mathbf{n}_e = 2\mathbf{Re}\left\{\sum_{k=1}^{4} \mathbf{U}_k^{\mathrm{T}}(\mathbf{A}_7 + p_k\mathbf{A}_8)F_k'(z_k)\right\}\mathbf{n}_e, \tag{48}$$

where

$$\mathbf{K}_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{K}_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{n}_\sigma = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{n}_e = \begin{bmatrix} 0 \\ 0 \\ \cos\theta \\ \sin\theta \end{bmatrix},$$

$$\mathbf{A}_5 = \begin{bmatrix} c_{15} & c_{14} & c_{13} & e_{31} \\ c_{56} & c_{46} & c_{36} & e_{36} \\ c_{55} & c_{45} & c_{35} & e_{35} \\ e_{15} & e_{14} & e_{13} & -\kappa_{13} \end{bmatrix}, \qquad \mathbf{A}_6 = \begin{bmatrix} c_{56} & c_{46} & c_{36} & e_{36} \\ c_{25} & c_{24} & c_{23} & e_{32} \\ c_{45} & c_{44} & c_{34} & e_{34} \\ e_{25} & e_{24} & e_{23} & -\kappa_{23} \end{bmatrix},$$

$$\mathbf{A}_7 = \begin{bmatrix} 0 & 0 & e_{11} & e_{21} \\ 0 & 0 & e_{16} & e_{26} \\ 0 & 0 & e_{15} & e_{25} \\ 0 & 0 & -\kappa_{11} & -\kappa_{12} \end{bmatrix}, \qquad \mathbf{A}_8 = \begin{bmatrix} 0 & 0 & e_{16} & e_{26} \\ 0 & 0 & e_{12} & e_{22} \\ 0 & 0 & e_{14} & e_{24} \\ 0 & 0 & -\kappa_{12} & -\kappa_{22} \end{bmatrix}.$$

θ is the angle between the x_1-axis and the outward normal to the boundary of the hole, measured counter-clockwise.

The mapping functions that map the exterior of an ellipse

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, \quad x_1 = a\cos\theta, \quad x_2 = b\sin\theta \tag{49}$$

for all z_k onto the exterior of a unit circle in the ξ_k planes are

$$z_k = m_k\xi_k + \bar{m}_k\xi_k^{-1}, \tag{50}$$

where

$$m_k = (a - \mathrm{i}p_k b)/2, \qquad \bar{m}_k = (a + \mathrm{i}p_k b)/2.$$

The inverse relations are

$$\xi_k = \frac{z_k + (z_k^2 - 4m_k\bar{m}_k)^{1/2}}{2m_k}. \tag{51}$$

For the auxiliary problem the electromechanical boundary conditions on the contour of the hole are

$$t_1 = -\sigma_0 \cos\theta, \quad t_2 = t_3 = 0, \quad D_n = -\sigma_0(d_{11}\cos\theta + d_{21}\sin\theta). \tag{52}$$

Imposing the boundary conditions on Eqs. (45)–(48), noticing that $\xi = \xi_k = \mathrm{e}^{\mathrm{i}\theta}$ on the unit circle, and

$$F_k'(z) = F_k'(\xi_k)\frac{\mathrm{d}\xi_k}{\mathrm{d}z_k} = F_k'(\xi_k)\frac{\xi_k}{m_k\xi_k - \bar{m}_k\xi_k^{-1}}, \tag{53}$$

multiplying both sides of the resulting equations by $(2\pi\mathrm{i})^{-1}\mathrm{d}\xi/(\xi - z)$ and integrating them around the unit circle clockwise, one obtains by using the Cauchy integral formula the following equations

$$\sum_{k=1}^{4} p_k \mathbf{U}_k^{\mathrm{T}}(\mathbf{A}_1^{\mathrm{T}} + p_k\mathbf{C}_{22})\boldsymbol{\eta}_1 F_k'(z) = -\frac{\sigma_0}{\pi}\int_0^{2\pi}\frac{\cos\theta}{\mathrm{e}^{\mathrm{i}\theta} - z}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta, \tag{54}$$

$$\sum_{k=1}^{4} \mathbf{U}_k^{\mathrm{T}}(\mathbf{A}_1^{\mathrm{T}} + p_k\mathbf{C}_{22})\boldsymbol{\eta}_1 F_k'(z) = 0, \tag{55}$$

$$\sum_{k=1}^{4} \mathbf{U}_k^{\mathrm{T}}(\mathbf{A}_5 + p_k\mathbf{A}_6)\boldsymbol{\eta}_1 F_k'(z) = 0, \tag{56}$$

$$\sum_{k=1}^{4} \mathbf{U}_k^{\mathrm{T}}(\mathbf{A}_7 + p_k\mathbf{A}_8)\boldsymbol{\eta}_2 F_k'(z) = \frac{\sigma_0}{\pi}\int_0^{2\pi}\frac{(d_{11}\cos\theta + d_{21}\sin\theta)}{\mathrm{e}^{\mathrm{i}\theta} - z}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta \tag{57}$$

for the four unknown functions $F_k'(z)$ $(k = 1, 2, 3, 4)$, where

$$\boldsymbol{\eta}_1 = \frac{z}{(m_k z - \overline{m}_k z^{-1})}\begin{bmatrix} z^{-1} + z \\ \mathrm{i}(z^{-1} - z) \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{\eta}_2 = \frac{z}{(m_k z - \overline{m}_k z^{-1})}\begin{bmatrix} 0 \\ 0 \\ z^{-1} + z \\ \mathrm{i}(z^{-1} - z) \end{bmatrix}.$$

On determining $F_k'(z)$ from Eqs. (54)–(57) and replacing z by z_k, the $F_k'(z_k)$ for the auxiliary problem are obtained. Superposition of the auxiliary solution and the infinite plate solution leads to the electromechanical field in the infinite piezoelectric plate with an elliptical hole as follows:

$$\boldsymbol{\tau}_1 = \sigma_0[0 \quad 1 \quad 0 \quad d_{11} \quad d_{31}]^{\mathrm{T}} + 2\mathrm{Re}\left\{\sum_{k=1}^{4}(\mathrm{A}_4 + p_k\mathbf{C}_{12})\mathbf{U}_k F_k'(z_k)\right\}, \tag{58}$$

$$\boldsymbol{\tau}_2 = \sigma_0[0 \quad 0 \quad 0 \quad d_{21}]^{\mathrm{T}} + 2\mathrm{Re}\left\{\sum_{k=1}^{4}(\mathbf{A}_1 + p_k\mathbf{C}_{22})\mathbf{U}_k F_k'(z_k)\right\}. \tag{59}$$

In closing, we remark that the solution of this problem for anisotropic elastic materials (Lekhnitskii, 1968) is obtained by setting $d_{ij} = e_{ij} = 0$. The notch problems can be solved as well by means of Laurent's series of complex variables as exemplified in Lekhnitskii (1981) for the elastic materials.

6. Concluding remarks

We have shown that the state space formalism is a simple and elegant approach to electrothermoelastic analysis of a piezoelectric body. Within the state space framework it is possible to derive the solution of piezothermoelasticity from its elastic counterpart by a replacement of the corresponding matrices. In case the boundary conditions for the piezoelectric material and for the anisotropic elastic material are in the same matrix form, the piezoelectric solution may be written out from the elastic solution. It is conceivable that piezothermoelasticity in the cylindrical coordinate system can be formulated in the same context following the state space formalism for elasticity with cylindrical anisotropy (Tarn, 2002b). Exact solutions for solid and hollow cylinders of piezoelectric materials with cylindrically anisotropy have recently been obtained (Tarn, 2001, 2002c) by an explicit formulation. The solution would be much simpler based on the present formalism.

In the present work we have confined attention to the static responses. In sensing and actuating applications of piezoelectric materials, vibrations and high frequency responses are of main concern. When dealing with elastodynamics of piezoelectric materials, albeit the clarity and simplicity of the formalism, the transient solution in three dimensions remains formidable. Nevertheless, solutions to the state equation for problems such as thickness vibrations, steady state vibrations of piezoelectric plates, fundamental standing waves, appear to be attainable. By far the great challenge lies in the three-dimensional analysis of transient responses. We shall continue the pursuit and develop the formalism along such lines.

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